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Fluid Dynamic Limits of Kinetic Equations. I. Formal Derivations

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The connection between kinetic theory and the macroscopic equations of fluid dynamics is described. In particular, our results concerning the incompressible Navier–Stokes equations are based on a formal derivation in which limiting moments are carefully balanced rather than on a classical expansion such as those of Hilbert or Chapman–Enskog. The moment formalism shows that the limit leading to the incompressible Navier–Stokes equations, like that leading to the compressible Euler equations, is a natural one in kinetic theory and is contrasted with the systematics leading to the compressible Navier–Stokes equations. Some indications of the validity of these limits are given. More specifically, the connection between the DiPerna–Lions renormalized solution of the classical Boltzmann equation and the Leray solution of the Navier–Stokes equations is discussed.

KEY WORDS: Boltzmann equation; Chapman–Enskog expansion; incompressible Navier stokes equation; renormalized and weak solutions.

1. INTRODUCTION

This paper is devoted to the connection between kinetic theory and macroscopic fluid dynamics. Formal limits are systematically derived and some rigorous results are given concerning the validity of these limits. In order to do that, several scalings are introduced for standard kinetic equations of the form

$$\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} C(F_\varepsilon) \tag{1}$$

Here $F_\varepsilon(t, x, v)$ is a nonnegative function representing the density of

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particles with position x and velocity v in the single-particle phase space $R_x^3 \times R_v^3$ at time t. The interaction of particles through collisions is modeled by the operator $C(F)$; this operator acts only on the variable v and is generally nonlinear. The operator will be kept abstract in Sections 1–4, being defined only by properties given below. This emphasizes a universality that embraces the Boltzmann, Fokker–Planck, and BGK forms for $C(F)$ (ref. 5) and allows the modeling of multiple-particle collisions. In Section 5 the classical Boltzmann form of the operator will be considered.

The connection between kinetic and macroscopic fluid dynamics results from two types of properties of the collision operator:

(i) Conservation properties and an entropy relation that implies that the equilibria are Maxwellian distributions for the zeroth-order limit.

(ii) The derivative of $C(F)$ satisfies a formal Fredholm alternative with a kernel related to the conservation properties of (i).

The macroscopic limits are obtained when the fluid becomes dense enough that particles undergo many collisions over the scales of interest. This situation is described by the introduction of a small parameter ε, called the Knudsen number, that represents the ratio of the mean free path of particles between collisions to some characteristic length of the flow (e.g., the size of an obstacle). Properties (i) are sufficient to derive the compressible Euler equations from Eq. (1); in Section 2 this will be done assuming a formally consistent convergence for the fluid dynamical moments and entropy of the solutions of the kinetic equation (1) (Theorem I). The compressible Euler equations also arise as the leading-order dynamics from a systematic expansion of F in ε (the Chapman–Enskog or Hilbert expansion described briefly in Section 3). Properties (ii) are used to obtain Navier–Stokes equations; they depend on a more detailed knowledge of the collision operator. The compressible Navier–Stokes equations arise as corrections to those of Euler at the next order in the Chapman–Enskog expansion; this is done in Section 3. Strong hypotheses are needed on the regularity of solutions of the compressible Navier–Stokes equations in order to make sense of these expansions (Theorem II).

In a compressible fluid one also introduces the Mach number Ma, which is the ratio of the bulk velocity to the sound speed, and the Reynolds number Re, which is a dimensionless reciprocal viscosity of the fluid. These numbers[1,2,12] are related by the formula

$$\varepsilon = \frac{Ma}{Re} \tag{2}$$

It is clear from this relation that when ε goes to zero, to obtain a fluid dynamical limit with a finite Reynolds number, the Mach number must vanish, too. This is the incompressible limit. This fact was already observed by Sone[15] in 1969 at the level of formal expansion for the steady solution. It was used by Sone and several of his co-workers for the study of the Knudsen layer (cf. for instance ref. 16). The incompressible limit is also the only regime where global weak solutions of fluid dynamic equations are known to exist. The relation between the global weak solutions of the incompressible Navier–Stokes equations due to Leray[11] and the renormalized solutions of the Boltzmann equation introduced by DiPerna and Lions[8] is the subject of our forthcoming companion paper.[2] The derivation of the incompressible Navier–Stokes equations presented in Section 4 (Theorem III) provides the framework for those results. Like Theorem I, it only assumes a formally consistent convergence for the fluid dynamical moments and entropy of the solutions of the kinetic equation (1). Some indications are given in Section 5 about the proof of the assumptions made in Section 4.

Related results have been obtained simultaneously by De Masi *et al.*[7] They prove the validity of a formal expansion over the time interval during which a solution of the incompressible Navier–Stokes equations remain smooth (cf. Remark 5 of Section 6). Due to his influence on the subject, we dedicate this paper to Joel Lebowitz on the occasion of his 60th birthday.

2. THE COMPRESSIBLE EULER LIMIT

In this section the integral of any scalar- or vector-valued function $f(v)$ with respect to the variable v will be denoted by $\langle f \rangle$,

$$\langle f \rangle = \int f(v)\, dv \tag{3}$$

The operator C is assumed to satisfy the conservation properties

$$\langle C(F) \rangle = 0, \qquad \langle vC(F) \rangle = 0, \qquad \langle |v|^2 C(F) \rangle = 0 \tag{4}$$

These relations represent the physical laws of mass, momentum, and energy conservation during collisions and imply the local conservation laws

$$\begin{aligned} \partial_t \langle F \rangle + \nabla_x \cdot \langle vF \rangle &= 0 \\ \partial_t \langle vF \rangle + \nabla_x \cdot \langle v \otimes vF \rangle &= 0 \\ \partial_t \langle \tfrac{1}{2}|v|^2 F \rangle + \nabla_x \cdot \langle v \tfrac{1}{2}|v|^2 F \rangle &= 0 \end{aligned} \tag{5}$$

Additionally, $C(F)$ is assumed to have the property that the quantity $\langle C(F) \log F\rangle$ is nonpositive. This is the entropy dissipation rate and implies the local entropy inequality

$$\partial_t \langle F \log F\rangle + \nabla_x \cdot \langle vF \log F\rangle = \langle C(F) \log F\rangle \leqslant 0 \tag{6}$$

Finally, the equilibria of $C(F)$ are assumed to be characterized by the vanishing of the entropy dissipation rate and given by the class of Maxwellian distributions, i.e., those of the form

$$F = \frac{\rho}{(2\pi\theta)^{3/2}} \exp\left(-\frac{1}{2}\frac{|v-u|^2}{\theta}\right) \tag{7}$$

More precisely, for every nonnegative measurable function F the following properties are equivalent:

$$\begin{aligned} &\text{(i)} \quad C(F) = 0, \\ &\text{(ii)} \quad \langle C(F) \log F\rangle = 0 \\ &\text{(ii)} \quad F \text{ is a Maxwellian with the form (7)} \end{aligned} \tag{8}$$

These assumptions about $C(F)$ merely abstract some of the consequences of Boltzmann's celebrated H-theorem.

The parameters ρ, u, and θ introduced in the right side of (7) are related to the fluid dynamic moments giving the mass, momentum, and energy densities:

$$\langle F\rangle = \rho, \qquad \langle vF\rangle = \rho u, \qquad \langle \tfrac{1}{2}|v|^2 F\rangle = \rho(\tfrac{1}{2}|u|^2 + \tfrac{3}{2}\theta)$$

They are called, respectively, the (mass) density, velocity, and temperature of the fluid. In the compressible Euler limit these variables are shown to satisfy the system of compressible Euler equations [(14) below].

The main obstruction to proving the validity of this fluid dynamical limit is the fact that solutions of the compressible Euler equations generally become singular after a finite time.[14] Therefore any global (in time) convergence proof cannot rely on uniform regularity estimates. The only reasonable assumptions would be that the limiting distribution exists and that the relevant moments converge pointwise. With this hypothesis, it is shown that the above assumptions regarding $C(F)$ imply that the fluid dynamic moments of solutions converge to a solution of the Euler equations that satisfies the macroscopic entropy inequality.

Theorem I. Given a collision operator C with properties (i), let $F_\varepsilon(t, x, v)$ be a sequence of nonnegative solutions of the equation

$$\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} C(F_\varepsilon) \tag{9}$$

such that as ε goes to zero, F_ε converges almost everywhere to a nonnegative function F. Moreover, assume that the moments

$$\langle F_\varepsilon \rangle, \quad \langle vF_\varepsilon \rangle, \quad \langle v \otimes vF_\varepsilon \rangle, \quad \langle v|v|^2 F_\varepsilon \rangle$$

converge in the sense of distributions to the corresponding moments

$$\langle F \rangle, \quad \langle vF \rangle, \quad \langle v \otimes vF \rangle, \quad \langle v|v|^2 F \rangle$$

the entropy densities and fluxes converge in the sense of distributions according to

$$\lim_{\varepsilon \to 0} \langle F_\varepsilon \log F_\varepsilon \rangle = \langle F \log F \rangle, \qquad \lim_{\varepsilon \to 0} \langle vF_\varepsilon \log F_\varepsilon \rangle = \langle vF \log F \rangle$$

while the entropy dissipation rates satisfy

$$\limsup_{\varepsilon \to 0} \langle C(F_\varepsilon) \log F_\varepsilon \rangle \leqslant \langle C(F) \log F \rangle$$

Then the limit $F(t, x, v)$ is a Maxwellian distribution,

$$F(t, x, v) = \frac{\rho(t, x)}{[2\pi\theta(t, x)]^{3/2}} \exp\left(-\frac{1}{2} \frac{|v - u(t, x)|^2}{\theta(t, x)} \right) \tag{10}$$

where the functions ρ, u, and θ solve the compressible Euler equations,

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot (\rho u) &= 0 \\ \partial_t(\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x(\rho\theta) &= 0 \\ \partial_t(\rho(\tfrac{1}{2}|u|^2 + \tfrac{3}{2}\theta)) + \nabla_x \cdot (\rho u(\tfrac{1}{2}|u|^2 + \tfrac{5}{2}\theta)) &= 0 \end{aligned} \tag{11}$$

and satisfy the entropy inequality,

$$\partial_t \left(\rho \log \left(\frac{\rho^{2/3}}{\theta} \right) \right) + \nabla_x \cdot \left(\rho u \log \left(\frac{\rho^{2/3}}{\theta} \right) \right) \leqslant 0 \tag{12}$$

Proof. Multiplying (9) by $\varepsilon(1 + \log F_\varepsilon)$ and integrating over v gives the entropy relation

$$\varepsilon(\partial_t \langle F_\varepsilon \log F_\varepsilon \rangle + \nabla_x \cdot \langle vF_\varepsilon \log F_\varepsilon \rangle) = \langle C(F_\varepsilon) \log F_\varepsilon \rangle \tag{13}$$

Letting ε go to zero in (13) and using the convergence assumptions of the theorem regarding the entropic quantities shows that the limiting distribution F must satisfy

$$0 \leqslant \limsup_{\varepsilon \to 0} \langle C(F_\varepsilon) \log F_\varepsilon \rangle \leqslant \langle C(F) \log F \rangle \tag{14}$$

But the entropy dissipation rate of $C(F)$ is nonpositive by assumption, so (14) implies $\langle C(F) \log F \rangle = 0$. The characterization of equilibria (8) then gives that for almost every (t, x) the distribution F is a solution of the equation $C(F) = 0$ and is a Maxwellian distribution with the form (10).

The system of local conservation laws

$$\begin{aligned} \partial_t \langle F_\varepsilon \rangle + \nabla_x \cdot \langle v F_\varepsilon \rangle &= 0 \\ \partial_t \langle v F_\varepsilon \rangle + \nabla_x \cdot \langle v \otimes v F_\varepsilon \rangle &= 0 \\ \partial_t \langle \tfrac{1}{2} |v|^2 F_\varepsilon \rangle + \nabla_x \cdot \langle v \tfrac{1}{2} |v|^2 F_\varepsilon \rangle &= 0 \end{aligned} \tag{15}$$

is not closed. Each of these equations for the determination of the time derivative of a moment involves the knowledge of a higher order moment. However, if the convergence assumptions of the theorem regarding these moments are used, one can pass to the limit of ε going to zero and replace F_ε by F, as given by (10), in these equations. A system of five equations for the five unknowns $\{\rho, u_1, u_2, u_3, \theta\}$ is obtained which is the compressible Euler system (11).

Finally, utilizing the entropy dissipation property

$$\langle C(F_\varepsilon) \log F_\varepsilon \rangle \leqslant 0 \tag{16}$$

we find that Eq. (9) leads to the inequality

$$\partial_t \langle F_\varepsilon \log F_\varepsilon \rangle + \nabla_x \cdot \langle v F_\varepsilon \log F_\varepsilon \rangle \leqslant 0 \tag{17}$$

Once again using the convergence hypothesis of the theorem regarding the entropy densities and fluxes along with the form of F given by (10), this inequality gives the classical entropy inequality (12).

Remark 1. The above arguement shows that any type of equation of the form (9) leads to the compressible Euler equations with a pressure p given by the ideal gas law $p = \rho\theta$ and an internal energy of $\frac{3}{2}\rho\theta$ (corresponding to a $\frac{5}{3}$-law perfect gas). This is a consequence of the fact that the kinetic equation considered here describes a monoatomic fluid in a three-dimensional domain. Other equations of state may be obtained by introducing additional degrees of freedom that take into account the rotational and vibrational modes of the particles.

3. THE COMPRESSIBLE NAVIER–STOKES LIMIT

As has been noticed above, the form of the limiting Euler equation is independent of the choice of the collision operator C within the class of operators satisfying the conservation and the entropy properties. The choice of the collision operator appears at the macroscopic level only in the construction of the Navier–Stokes limit. The compressible Navier–Stokes equations are obtained by the classical Chapman–Enskog expansion. To compare this approach with the situation leading to the incompressible Navier–Stokes equation, a short description of this approach is given below.

Given (ρ, u, θ), denote the corresponding Maxwellian distribution by

$$M_{(\rho,u,\theta)} = \frac{\rho}{(2\pi\theta)^{3/2}} \exp\left(-\frac{1}{2}\frac{|v-u|^2}{\theta}\right) \tag{18}$$

The subscript (ρ, u, θ) will often be omitted when it is convenient. Introduce the Hilbert space L^2_M defined by the scalar product

$$(f\,|\,g)_M = \langle fg \rangle_M = \int f(v)\,g(v)\,M(v)\,dv \tag{19}$$

Denote by L and Q the first two Fréchet derivatives of the operator $G \mapsto M^{-1}C(MG)$ at $G = 1$:

$$L(g) = \frac{1}{M} DC(M)\cdot(Mg), \qquad Q(g,g) = \frac{1}{M} D^2C(M):(Mg \vee Mg) \tag{20}$$

where $\vee$ is the usual symmetric tensor product over functions of v. Taylor's formula then gives

$$\frac{1}{M} C(M(1+\varepsilon g)) = \varepsilon L(g) + \varepsilon^2 \tfrac{1}{2} Q(g,g) + O(\varepsilon^3) \tag{21}$$

In general the operator L is not bounded. However, it is naturally defined as a linear unbounded operator in the space L^2_M. This operator is assumed to be self-adjoint and to satisfy a Fredholm alternative with a five-dimensional kernel spanned by the functions $\{1, v_1, v_2, v_3, |v|^2\}$. The fact that the kernel must contain at least these vectors follows from deriving the relation $C(M) = 0$ over all Maxwellian distributions (7). Examining the second variation of the entropy dissipation rate at M shows that L has a nonpositive Hermitian form.

Denote by V, $A = \{A_i\}$ and $B = \{B_{ij}\}$ the following vectors and tensors:

$$V = \frac{v-u}{\sqrt{2}}, \qquad A(V) = (\tfrac{1}{2}|V|^2 - \tfrac{5}{2})V, \qquad B(V) = V \otimes V - \tfrac{1}{3}|V|^2 I$$

By symmetry, the functions A_i and B_{ij} are orthogonal to the kernel of L; therefore the equations

$$L(A') = A, \qquad L(B') = B \tag{22}$$

have unique solutions in $\mathrm{Ker}(L)^{\perp}$. Assume (this would be a consequence of rotational invariance for the collision operator) that these solutions are given by the formulas

$$A'(V) = -\alpha(\rho, \theta, |V|)\, A(V), \qquad B'(V) = -\beta(\rho, \theta, |V|)\, B(V) \tag{23}$$

where α and β are positive functions depending on ρ, θ, and $|V|$. If $C(F)$ is homogeneous of degree two (for example, if it is quadratic), then a simple scaling shows that the ρ dependence of α and β is just proportional to ρ^{-1}.

A function $H_\varepsilon(t, x, v)$ is said to be an approximate solution of order p to the kinetic equation (1) if

$$\partial_t H_\varepsilon + v \cdot \nabla_x H_\varepsilon = \frac{1}{\varepsilon} C(H_\varepsilon) + O(\varepsilon^p) \tag{24}$$

where $O(\varepsilon^p)$ denotes a term bounded by ε^p in some convenient norm. An approximate solution of order 2 will be constructed in the form

$$H_\varepsilon = M_\varepsilon(1 + \varepsilon g_\varepsilon + \varepsilon^2 w_\varepsilon) \tag{25}$$

where $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)$ solve the compressible Navier–Stokes equations with dissipation of the order ε (denoted $\mathrm{CNSE}_\varepsilon$):

$$\begin{aligned}
&\partial_t \rho_\varepsilon + \nabla_x \cdot (\rho_\varepsilon u_\varepsilon) = 0\\
&\rho_\varepsilon(\partial_t + u_\varepsilon \cdot \nabla_x) u_\varepsilon + \nabla_x(\rho_\varepsilon \theta_\varepsilon) = \varepsilon \nabla_x \cdot [\mu_\varepsilon \sigma(u_\varepsilon)]\\
&\tfrac{3}{2}\rho_\varepsilon(\partial_t + u_\varepsilon \cdot \nabla_x)\theta_\varepsilon + \rho_\varepsilon \theta_\varepsilon \nabla_x \cdot u_\varepsilon = \varepsilon \tfrac{1}{2}\mu_\varepsilon \sigma(u_\varepsilon) : \sigma(u_\varepsilon) + \varepsilon \nabla_x \cdot [\kappa_\varepsilon \nabla_x \theta_\varepsilon]
\end{aligned} \tag{26}$$

In these equations $\sigma(u)$ denotes the strain-rate tensor given by

$$\sigma_{ij}(u) = (u^i_{x_j} + u^j_{x_i}) - \tfrac{2}{3}\nabla_x \cdot u\,\delta_{ij}$$

while the viscosity $\mu_\varepsilon = \mu(\rho_\varepsilon, \theta_\varepsilon)$ and the thermal diffusivity $\kappa_\varepsilon = \kappa(\rho_\varepsilon, \theta_\varepsilon)$ are defined by the relations

$$\begin{aligned}
\mu(\rho, \theta) &= \frac{1}{10}\theta \langle \beta(\rho, \theta, |V|)\,|B(V)|^2 \rangle_M\\
&= \frac{2}{15}\rho\theta \frac{1}{(2\pi)^{1/2}} \int_0^\infty \beta(\rho, \theta, r) r^6 e^{-r^2/2}\, dr\\
\kappa(\rho, \theta) &= \frac{1}{3}\theta \langle \alpha(\rho, \theta, |V|)\,|A(V)|^2 \rangle_M\\
&= \frac{1}{6}\rho\theta \frac{1}{(2\pi)^{1/2}} \int_0^\infty \alpha(\rho, \theta, r)(r^2 - 5)^2 r^4 e^{-r^2/2}\, dr
\end{aligned} \tag{27}$$

Notice that in the case where $C(F)$ is homogeneous of degree two, the left sides become independent of ρ; this is why classical expressions for the viscosity and thermal diffusivity depend only on θ.

The Chapman–Enskog derivation can be formulated according to the following.

Theorem II. Assume that $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)$ solve the CNSE$_\varepsilon$ with the viscosity $\mu(\rho, \theta)$ and thermal diffusivity $\kappa(\rho, \theta)$ given by (27). Then there exist g_ε and w_ε in $\mathrm{Ker}(L)^\perp$ such that H_ε, given by (25), is an approximate solution of order 2 to Eq. (1). Moreover, g_ε is given by the formula

$$g_\varepsilon = -\frac{1}{2}\beta(\rho_\varepsilon, \theta_\varepsilon, |V|)\, B(V):\sigma(u_\varepsilon) - \alpha(\rho_\varepsilon, \theta_\varepsilon, |V|)\frac{A(V)\cdot\nabla_x\theta_\varepsilon}{\sqrt{\theta_\varepsilon}} \qquad (28)$$

Proof. In the computation below the subscript ε is omitted. Setting the form (25) for an approximate solution of order two into (24) yields the formula

$$\begin{aligned}\frac{(\partial_t + v\cdot\nabla_x)M}{M} &+ \varepsilon\frac{(\partial_t + v\cdot\nabla_x)(Mg)}{M}\\ &= L(g) + \varepsilon\left(L(w) + \frac{1}{2}Q(g, g)\right) + O(\varepsilon^2)\end{aligned} \qquad (29)$$

The $O(\varepsilon^2)$ term is the remainder from the Taylor expansion of $M^{-1}C(M(1+\varepsilon q+\varepsilon^2 w))$ about $\varepsilon = 0$ and is therefore in $\mathrm{Ker}(L)^\perp$. A direct derivation of (18) gives the formulas

$$\partial_u M = V\frac{1}{\sqrt{\theta}}M, \qquad \partial_\theta M = \left(\frac{1}{2}|V|^2 - \frac{3}{2}\right)\frac{1}{\theta}M$$

Utilizing these shows that the contribution of the first term on the left side of (29) is given by the formula

$$\begin{aligned}\frac{(\partial_t + v\cdot\nabla_x)M}{M} &= \frac{(\partial_t + v\cdot\nabla_x)\rho}{\rho} + V\cdot\frac{(\partial_t + v\cdot\nabla_x)u}{\sqrt{\theta}}\\ &\quad + \left(\frac{1}{2}|V|^2 - \frac{3}{2}\right)\frac{(\partial_t + v\cdot\nabla_x)\theta}{\theta}\end{aligned} \qquad (30)$$

The CNSE$_\varepsilon$(27) are used to replace the time derivatives of the functions ρ, u, and θ by expressions involving only spatial derivatives. This introduces terms of order ε, corresponding to the right side of the second and third equations of (27), into (30):

$$\frac{(\partial_t + v\cdot\nabla_x)M}{M} = \frac{1}{2}B(V):\sigma(u) + A(V)\cdot\frac{\nabla_x\theta}{\sqrt{\theta}} + \varepsilon R \tag{31}$$

where

$$R = V\cdot\frac{(\nabla_x\cdot[\mu\sigma(u)])}{\rho\sqrt{\theta}} + \left(\frac{1}{3}|V|^2 - 1\right)\frac{\frac{1}{2}\mu\sigma(u):\sigma(u) + \nabla_x\cdot[\kappa\nabla_x\theta]}{\rho\theta} \tag{32}$$

From (29) and (31) it follows that the term of order one with respect to ε has to be given by the formula (28). To complete the proof, one must show the existence of a function w that cancels the term of order one in (29). This amounts to proving the existence of a solution to the equation

$$L(\omega) = R + \frac{(\partial_t + v\cdot\nabla_x)(Mg)}{M} - \frac{1}{2}Q(g, g) \tag{33}$$

Such a solution exists if and only if the right side of (33) is orthogonal to the kernel of L. These orthogonality relations are already satisfied by the terms $Q(g,g)$ and $M^{-1}\partial_t(Mg)$, so all that is left to compute are the remaining terms. The inner products of 1, $v-u$, and $\frac{1}{2}|v-u|^2$ with R are easily computed from (32) and are given by

$$\begin{aligned} &\langle R\rangle_M = 0, \qquad \langle (v-u)R\rangle_M = \nabla_x\cdot[\mu\sigma(u)] \\ &\langle \tfrac{1}{2}|v-u|^2 R\rangle_M = \tfrac{1}{2}\mu\sigma(u):\,\sigma(u) + \nabla_x\cdot[\kappa\nabla_x\theta] \end{aligned} \tag{34}$$

When computing the same inner products for the $M^{-1}v\cdot\nabla_x(Mg)$ term, the orthogonality of g to the Ker(L) implies

$$\begin{aligned} &\left\langle\frac{v\cdot\nabla_x(Mg)}{M}\right\rangle_M = 0, \qquad \left\langle (v-u)\frac{v\cdot\nabla_x(Mg)}{M}\right\rangle_M = \nabla_x\cdot[\theta\langle Bg\rangle_M] \\ &\left\langle \frac{1}{2}|v-u|^2\frac{v\cdot\nabla_x(Mg)}{M}\right\rangle_M = \nabla_x\cdot[\theta^{3/2}\langle Ag\rangle_M] + \frac{1}{2}\theta\langle Bg\rangle_M:\,\sigma(u) \end{aligned} \tag{35}$$

With the explicit form of g given by (28), one finds that

$$\begin{aligned} \theta^{3/2}\langle Ag\rangle_M &= -\theta\langle\alpha A\otimes A\rangle_M\cdot\nabla_x\theta = -\tfrac{1}{12}\theta\langle\alpha(|V|^2-5)^2|V|^2\rangle_M\nabla_x\theta \\ \theta\langle Bg\rangle_M &= -\tfrac{1}{2}\theta\langle\beta B\otimes B\rangle_M:\sigma(u) = -\tfrac{1}{15}\theta\langle\beta|V|^4\rangle_M\,\sigma(u) \end{aligned} \tag{36}$$

Putting the right side of (36) into (35) and comparing the result with (34) shows that the solvability condition for (33) is satisfied if and only if $\mu(\rho, \theta)$

and $\kappa(\rho, \theta)$ are given by formula (27). This completes the proof of Theorem II.

Remark 2. Let F_ε be a solution of the kinetic equation that coincides with a local Maxwellian at $t=0$. Let $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)$ be the solution of the CNSE_ε with initial data equal to the corresponding moments of $F_\varepsilon(0, x, v)$. Then it follows from the above computation that the expression given by

$$H_\varepsilon = \frac{1}{(2\pi\theta_\varepsilon(t, x))^{3/2}} \exp\left(-\frac{1}{2}\frac{|v-u_\varepsilon(t, x)|^2}{\theta_\varepsilon(t, x)}\right)(\rho_\varepsilon(t, x) + \varepsilon g_\varepsilon + \varepsilon^2 w_\varepsilon)$$

is an approximation of order 2 of F_ε. Since $M_\varepsilon g_\varepsilon$ is orthogonal to the functions 1, v, $|v|^2$, the quantities ρ_ε, $\rho_\varepsilon u_\varepsilon$, and $\rho_\varepsilon(\frac{1}{2}|u_\varepsilon|^2 + \frac{3}{2}\theta_\varepsilon)$ provide approximations of order two to the corresponding moments of F_ε. In fact, this observation was used to do the Chapman–Enskog derivation by the so-called projection method.[4]

4. THE INCOMPRESSIBLE NAVIER–STOKES LIMIT

The purpose of this section is to construct a connection between the kinetic equation and the incompressible Navier–Stokes equations. As in the previous section, this will describe the range of parameters for which the incompressible Navier–Stokes equations provide a good approximation to the solution of the kinetic equation. However, in this case the connection is drawn between the kinetic equation and macroscopic fluid dynamic equations with a finite Reynolds number. It is clear form formula (2), $\varepsilon = Ma/Re$, that in order to obtain a fluid dynamic regime (corresponding to a vanishing Knudsen number) with a finite Reynolds number, the Mach number must vanish.[1,12]

In order to realize distributions with a small Mach number, it is natural to consider them as perturbations about a given absolute Maxwellian (constant in space and time). By the proper choice of Galilean frame and dimensional units this absolute Maxwellian can be taken to have velocity equal to 0, and density and temperature equal to 1; it will be denoted by M. The initial data $F_\varepsilon(0, x, v)$ is assumed to be close to M, where the order of the distance will be measured with the Knudsen number. Furthermore, if the flow is to be incompressible, the kinetic energy of the flow in the acoustic model must be smaller than that in the rotational modes. Since the acoustic modes vary on a faster time scale than rotational modes, they may be suppressed by assuming that the initial data is consistent with motion on a slow time scale; this scale separation will also be measured with the Knudsen number.

This scaling is quantifed by the introduction of a small parameter ε

such that the time scale considered is of order ε^{-1}, the Knudsen number is of order ε^q, and the distance to the absolute Maxwellian M is of order ε^r, with q and r being greater than or equal to one. Thus, solutions F_ε to the equation

$$\varepsilon \partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon^q} C(F_\varepsilon) \tag{37}$$

are sought in the form

$$F_\varepsilon = M(1 + \varepsilon^r g_\varepsilon) \tag{38}$$

The basic case $r = q = 1$ is the unique scaling compatible with the usual incompressible Navier–Stokes equations.

The notation introduced in the previous section regarding the collision operator and its Fréchet derivatives is conserved, but here the Maxwellian M is absolute, so that L and Q no longer depend on the fluid variables.

Theorem III. Let $F_\varepsilon(t, x, v)$ be a sequence of nonnegative solutions to the scaled kinetic equation (37) such that, when it is written according to formula (38), the sequence g_ε converges in the sense of distributions and almost everywhere to a function g as ε goes to zero. Furthermore, assume the moments

$$\langle g_\varepsilon \rangle_M, \quad \langle v g_\varepsilon \rangle_M, \quad \langle v \otimes v g_\varepsilon \rangle_M, \quad \langle v |v|^2 g_\varepsilon \rangle_M$$
$$\langle L^{-1}(A(v)) \otimes v g_\varepsilon \rangle_M, \quad \langle L^{-1}(A(v))\, Q(g_\varepsilon, g_\varepsilon) \rangle_M$$
$$\langle L^{-1}(B(v)) \otimes v g_\varepsilon \rangle_M, \quad \langle L^{-1}(B(v))\, Q(g_\varepsilon, g_\varepsilon) \rangle_M$$

converge in $D'(R_t^+ \times R_x^3)$ to the corresponding moments

$$\langle g \rangle_M, \quad \langle v g \rangle_M, \quad \langle v \otimes v g \rangle_M, \quad \langle v |v|^2 g \rangle_M$$
$$\langle L^{-1}(A(v)) \otimes v g \rangle_M, \quad \langle L^{-1}(A(v))\, Q(g, g) \rangle_M$$
$$\langle L^{-1}(B(v)) \otimes v g \rangle_M, \quad \langle L^{-1}(B(v))\, Q(g, g))_M$$

and that all formally small terms in ε vanish. Then the limiting g has the form

$$g = \rho + v \cdot u + (\tfrac{1}{2}|v|^2 - \tfrac{3}{2})\theta \tag{39}$$

where the velocity u is divergence-free and the density and temperature fluctuations ρ and θ satisfy the Boussinesq relation

$$\nabla_x \cdot u = 0, \qquad \nabla_x(\rho + \theta) = 0 \tag{40}$$

Moreover, the functions ρ, u, and θ are weak solutions of the equations

$$\partial_t u + u\cdot\nabla_x u + \nabla_x p = \mu_* \Delta u, \quad \partial_t\theta + u\cdot\nabla_x\theta = \kappa_*\Delta\theta, \quad \text{if } r=1,\ q=1 \tag{41}$$

$$\partial_t u + \nabla_x p = \mu_* \Delta u, \quad \partial_t\theta = \kappa_*\Delta\theta, \quad \text{if } r>1,\ q=1 \tag{42}$$

$$\partial_t u + u\cdot\nabla_x u + \nabla_x p = 0, \quad \partial_t\theta + u\cdot\nabla_x\theta = 0, \quad \text{if } r=1,\ q>1 \tag{43}$$

$$\partial_t u + \nabla_x p = 0, \quad \partial_t\theta = 0, \quad \text{if } r>1,\ q>1 \tag{44}$$

In these equations the expressions μ_* and κ_* denote the function values $\mu(1, 1)$ and $\kappa(1, 1)$ obtained from (27) in the previous section.

Remark 3. Equation (44) is completely trivial; it corresponds to a situation where the initial fluctuations and the Knudsen number are too small to produce any evolution over the time scale selected. However, this limit would be nontrivial if it corresponded to a time scale on which an external potential force acts on the system.[1]

Proof. Setting (38) into (37) and Taylor expanding the collision operator gives

$$\varepsilon\partial_t g_\varepsilon + v\cdot\nabla_x g_\varepsilon = \frac{1}{\varepsilon^q} L(g_\varepsilon) + \varepsilon^{r-q}\frac{1}{2} Q(g_\varepsilon, g_\varepsilon) + O(\varepsilon^{2r} - q) \tag{45}$$

Multiplying this by ε^q, letting ε go to zero, and using the moment convergence assumption yields the relation

$$L(g) = 0 \tag{46}$$

This implies that g belongs to the kernel of L and thus can be written according to the formula (39).

The derivation of (40) starts from the equations for conservation of mass and momentum associated with (45):

$$\varepsilon\partial_t\langle g_\varepsilon\rangle_M + \nabla_x\cdot\langle v g_\varepsilon\rangle_M = 0 \tag{47}$$

$$\varepsilon\partial_t\langle v g_\varepsilon\rangle_M + \nabla_x\cdot\langle v\otimes v g_\varepsilon\rangle_M = 0 \tag{48}$$

Letting ε go to zero above (understanding the limit to be in the sense of distributions) gives the relations

$$\nabla_x\cdot\langle v g\rangle_M = 0, \qquad \nabla_x\cdot\langle v\otimes v g\rangle_M = 0$$

When g is replaced by the right side of (39) these become (40). A repetition of this argument starting from the equation for the conservation of energy again leads to the divergence-free velocity condition of (40).

The limiting momentum equation is obtained from

$$\partial_t \langle v g_\varepsilon \rangle_M + \frac{1}{\varepsilon} \nabla_x \cdot \langle v \otimes v g_\varepsilon \rangle_M = 0 \tag{49}$$

by first separating the flux tensor into its traceless and diagonal parts:

$$\partial_t \langle v g_\varepsilon \rangle_M + \frac{1}{\varepsilon} \nabla_x \cdot \left\langle \left(v \otimes v - \frac{1}{3} |v|^2 I \right) g_\varepsilon \right\rangle_M + \frac{1}{\varepsilon} \nabla_x \left\langle \frac{1}{3} |v|^2 g_\varepsilon \right\rangle_M = 0 \tag{50}$$

This is best thought of as being in the form

$$\partial_t \langle v g_\varepsilon \rangle_M + \frac{1}{\varepsilon} \nabla_x \cdot \langle B(v)\, g_\varepsilon \rangle_M + \nabla_x p_\varepsilon = 0 \tag{51}$$

where the pressure is given by $p_\varepsilon = \varepsilon^{-1} \langle \frac{1}{3} |v|^2 g_\varepsilon \rangle_M$. In the same spirit, the limiting temperature equation is obtained by combining the density and energy equations for (37) as

$$\partial_t \left\langle \left(\frac{1}{2} |v|^2 - \frac{5}{2} \right) g_\varepsilon \right\rangle_M + \frac{1}{\varepsilon} \nabla_x \cdot \langle A(v)\, g_\varepsilon \rangle_M = 0 \tag{52}$$

Utilization of the moment convergence assumption and the limiting form of g given by (39) provides the evaluation of the distribution limits

$$\begin{aligned} \lim_{\varepsilon \to 0} \partial_t \langle v g_\varepsilon \rangle_M &= \partial_t \langle v g \rangle_M = \partial_t u \\ \lim_{\varepsilon \to 0} \partial_t \langle (\tfrac{1}{2} |v|^2 - \frac{5}{2})\, g_\varepsilon \rangle_M &= \partial_t \langle (\tfrac{1}{2} |v|^2 - \tfrac{5}{2})\, g \rangle_M = \tfrac{5}{2} \partial_t \theta \end{aligned} \tag{53}$$

As is classical (since the contribution of Leray) in most treatments of the incompressible Navier–Stokes equations, the pressure term that appears on the right side of (51) will be eliminated upon integrating the equation against a divergence-free test function.

To complete the proof of Theorem III, the limits of the moments $\varepsilon^{-1} \langle A(v)\, g_\varepsilon \rangle_M$ in (52) and $\varepsilon^{-1} \langle B(v)\, g_\varepsilon \rangle_M$ in (51) have to be estimated. Start from the identities (recall that L is self-adjoint)

$$\langle A(v)\, g_\varepsilon \rangle_M = \langle L^{-1}(A(v))\, L(g_\varepsilon) \rangle_M, \quad \langle B(v)\, g_\varepsilon \rangle_M = \langle L^{-1}(B(v))\, L(g_\varepsilon) \rangle_M$$

and eliminate $L(g_\varepsilon)$ using Eq. (45),

$$\varepsilon \partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon = \frac{1}{\varepsilon^q} L(g_\varepsilon) + \varepsilon^{r-q} \frac{1}{2} Q(g_\varepsilon, g_\varepsilon) + O(\varepsilon^{2r-q})$$

The convergence assumptions of the theorem then imply that the limiting moments may be evaluated by

$$
\begin{aligned}
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \langle A(v)\, g_\varepsilon \rangle_M &= \lim_{\varepsilon \to 0} \varepsilon^{q-1} \nabla_x \cdot \langle v \otimes L^{-1}(A(v))\, g_\varepsilon \rangle_M \\
&\quad - \lim_{\varepsilon \to 0} \varepsilon^{r-1} \frac{1}{2} \langle L^{-1}(A(v))\, Q(g_\varepsilon, q_\varepsilon) \rangle_M \\
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \langle B(v)\, g_\varepsilon \rangle_M &= \lim_{\varepsilon \to 0} \varepsilon^{q-1} \nabla_x \cdot \langle v \otimes L^{-1}(B(v))\, g_\varepsilon \rangle_M \\
&\quad - \lim_{\varepsilon \to 0} \varepsilon^{r-1} \frac{1}{2} \langle L^{-1}(B(v))\, Q(g_\varepsilon, q_\varepsilon) \rangle_M
\end{aligned}
\tag{54}
$$

The neglected terms are formally $O(\varepsilon)$ and $O(\varepsilon^{2r-1})$. For $r > 1$ and $q > 1$ all the limits on the right side of (54) will vanish by the moment convergence assumptions; formula (44) is a direct consequence. For the case $r > q = 1$ one needs to compute the moments

$$
\nabla_x \cdot \langle v \otimes L^{-1}(A(v))\, g \rangle_M, \qquad \nabla_x \cdot \langle v \otimes L^{-1}(B(v))\, g \rangle_M
$$

For the case $q > r = 1$ one needs the computation of

$$
\langle L^{-1}(A(v))\, Q(g, g) \rangle_M, \qquad \langle L^{-1}(B(v))\, Q(g, g) \rangle_M
$$

For the case $r = q = 1$ the knowledge of all four of the above moments will be needed.

The limiting form (39) and the Boussinesq relation (40) imply that

$$
\begin{aligned}
\nabla_x \cdot \langle v \otimes L^{-1}(A(v))\, g \rangle_M &= \langle L^{-1}(A(v)) \otimes v(\tfrac{1}{2}|v|^2 - \tfrac{5}{2}) \rangle_M \cdot \nabla_x \theta \\
&= -\langle \alpha(1, 1, |v|)\, A(v) \otimes A(v) \rangle_M \cdot \nabla_x \theta
\end{aligned}
\tag{55}
$$

This expression gives the thermal diffusion term appearing in the second equation of systems (41) and (42). Even more directly, the limiting form (39) implies

$$
\begin{aligned}
\nabla_x \cdot \langle v \otimes L^{-1}(B(v)) g \rangle_M &= \langle L^{-1}(B(v)) \otimes (v \otimes v) \rangle_M : \nabla_x u \\
&= -\langle \beta(1, 1, |v|)\, B(v) \otimes B(v) \rangle_M : \nabla_x u
\end{aligned}
\tag{56}
$$

After applying a divergence, this expression gives the viscous term appearing in the first equation of systems (41) and (42).

Next, consider the moments $\langle L^{-1}(A(v))\, Q(g, g) \rangle_M$ and $\langle L^{-1}(B(v))\, Q(g, g) \rangle_M$; these may be evaluated by using the fact that $C(F)$ vanishes for all Maxwellians (18). The first and second differentials of $M_{(\rho,u,\theta)}$ computed at the point (1, 0, 1) are

$$dM = M(d\rho + v \cdot du + (\tfrac{1}{2}|v|^2 - \tfrac{3}{2})\, d\theta) \tag{57}$$

$$\begin{aligned} d^2M &= M(d\rho + v \cdot du + (\tfrac{1}{2}|v|^2 - \tfrac{3}{2})\, d\theta)^2 \\ &\quad + M(d^2\rho + v \cdot d^2u + (\tfrac{1}{2}|v|^2 - \tfrac{3}{2})\, d^2\theta) \end{aligned} \tag{58}$$

Comparison of (57) with the limiting form (39) shows that a correct choice of parameterization leads to $dM = M_g$ and $d^2M = Mg^2$. Twice deriving the formula that states Maxwellians are equilibria for the collision operator then gives

$$\begin{aligned} 0 = d^2C(M) &= D^2C(M):(dM \vee dM) + DC(M) \cdot d^2M \\ &= D^2C(M):(Mg \vee Mg) + DC(M) \cdot (Mg^2) \end{aligned} \tag{59}$$

Applying the definitions (20) of L and Q, this becomes simply

$$Q(g, g) = -L(g^2) \tag{60}$$

Using relation (60) and the self-adjointness of L, one finds the desired moments to be

$$\begin{aligned} &\langle L^{-1}(A(v))\, Q(g, g)\rangle_M \\ &\quad = -\langle L^{-1}(A(v))\, L(g^2)\rangle_M = -\langle A(v)\, g^2\rangle_M = -5u\theta \end{aligned} \tag{61}$$

$$\begin{aligned} &\langle L^{-1}(B(v))\, Q(g, g)\rangle_M \\ &\quad = -\langle L^{-1}(B(v))\, L(g^2)\rangle_M = -\langle B(v)\, g^2\rangle_M = -2B(u) \end{aligned} \tag{62}$$

Formula (61) gives the term $u \cdot \nabla_x \theta$ appearing in the second equation of the systems (41) and (43), while (62) gives the term $u \cdot \nabla_x u$ that appears in the first equation of the systems (41) and (43). The proof of Theorem II is now complete.

Remark 4. The second equations of (40)–(42) that describe the evolution of the temperature do not contain a viscous heating term $\frac{1}{2}\mu_* \sigma(u):\sigma(u)$ such as appears in the $\mathrm{CNSE}_\varepsilon$:

$$\begin{aligned} &\partial_t \rho_\varepsilon + \nabla_x \cdot (\rho_\varepsilon u_\varepsilon) = 0 \\ &\rho_\varepsilon(\partial_t + u_\varepsilon \cdot \nabla_x) u_\varepsilon + \nabla_x(\rho_\varepsilon \theta_\varepsilon) = \varepsilon \nabla_x \cdot [\mu_\varepsilon \sigma(u_\varepsilon)] \\ &\tfrac{3}{2}\rho_\varepsilon(\partial_t + u_\varepsilon \cdot \nabla_x)\theta_\varepsilon + \rho_\varepsilon \theta_\varepsilon \nabla_x \cdot u_\varepsilon = \varepsilon \tfrac{1}{2}\mu_\varepsilon \sigma(u_\varepsilon):\sigma(u_\varepsilon) + \varepsilon \nabla_x \cdot [\kappa_\varepsilon \nabla_x \theta_\varepsilon] \end{aligned} \tag{63}$$

This is consistent with the scaling used here when it is applied directly to the CNSE$_\varepsilon$ to derive the incompressible Navier–Stokes equations. More precisely, with the change of variables

$$\rho_\varepsilon = \rho_0 + \varepsilon\tilde{\rho}\left(t, \frac{x}{\varepsilon}\right), \qquad u_\varepsilon = \varepsilon\tilde{u}\left(t, \frac{x}{\varepsilon}\right), \qquad \theta_\varepsilon = \theta_0 + \varepsilon\tilde{\theta}\left(t, \frac{x}{\varepsilon}\right) \tag{64}$$

the system (40), (41) is obtained for $\tilde{\rho}(t, \tilde{x})$, $\tilde{u}(t, \tilde{x})$, and $\tilde{\theta}(t, \tilde{x})$ as ε vanishes. In this derivation every term of the last equation of (62) is of the order ε^2 except $\frac{1}{2}\mu_\varepsilon \sigma(u_\varepsilon):\sigma(u_\varepsilon)$, which is of order three. The viscous heating term would have appeared in the limiting temperature equation had the scaling in (64) been chosen with the density and temperature fluctuations of order ε^2.[3]

Remark 5. In the case where $q = r = 1$ a system is obtained that has some structure in common with a diffusion approximation. A formal expansion for q_ε, the solution of the equation

$$\varepsilon\partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon = \frac{1}{\varepsilon} L(g_\varepsilon) + \frac{1}{2} Q(g_\varepsilon, g_\varepsilon) + O(\varepsilon) \tag{65}$$

can be constructed in the form

$$g_\varepsilon = g^{(1)} + \varepsilon g^{(2)} + \varepsilon^2 g^{(3)} + \cdots \tag{66}$$

Inserting (66) into (65) and canceling terms through $O(\varepsilon)$ yields the system of equations

$$L(g^{(1)}) = 0 \tag{67}$$

$$L(g^{(2)}) = v \cdot \nabla_x g^{(1)} - \tfrac{1}{2} Q(g^{(1)}, g^{(1)}) \tag{68}$$

$$L(g^{(3)}) = \partial_t g^{(1)} + v \cdot \nabla_x g^{(2)} - O(1) \tag{69}$$

Equation (67) shows that $g^{(1)}$ is given [cf. (39)] by an expression of the form

$$g^{(1)} = \rho + v \cdot u + (\tfrac{1}{2}|v|^2 - \tfrac{3}{2})\theta \tag{70}$$

The solvability conditions for (68) are equivalent to the facts that u is divergence-free and that ρ and θ satisfy the Boussinesq relation. That satisfied, then $g^{(2)}$ is given by

$$\begin{aligned} g^{(2)} = & -\beta(1, 1, |v|)\, B(v):\nabla_x u - \alpha(1, 1, |v|)\, A(v) \cdot \nabla_x \theta \\ & + \tfrac{1}{2} B(v):(u \otimes u) + A(v) \cdot u\theta + (\tfrac{1}{8}|v|^4 - \tfrac{5}{4}|v|^2 + \tfrac{15}{8})\theta^2 \\ & + \rho^{(2)} + v \cdot u^{(2)} + (\tfrac{1}{2}|v|^2 - \tfrac{3}{2})\theta^{(2)} \end{aligned} \tag{71}$$

Finally, the solvability conditions for (69) given the two equations (41) for u and θ. This approach is related to the method of the previous section and to the work of De Masi, Esposito, and Lebowitz.

5. REMARKS CONCERNING THE PROOF OF THE FLUID DYNAMICAL LIMIT

In this section the collision operator is given by the classical Boltzmann formula,

$$C_B(F) = \iint_{R^3 \times S^2} (F(v_1')\, F(v') - F(v_1)\, F(v))\, b(v_1 - v, \omega)\, d\omega\, dv_1 \qquad (72)$$

where ω ranges over the unit sphere, v_1 over the three-dimensional velocity space, and $b(v_1 - v, \omega)$ is a smooth function; v' and v_1' are given in term of v, v_1, and ω by the classical relations of conservation of mass, momentum, and energy.[5,6]

Any proof concerning the fluid dynamical limit for a kinetic model will, as a by-product, give an existence proof for the corresponding macroscopic equation. However, up to now no new result has been obtained by this type of method. Uniform regularity estimates would likely be needed for obtaining the limit of the nonlinear term. These estimates, if they exist, must be sharp, because it is known (and is proven by Sideris[14] for a very general situation) that the solutions of the compressible nonlinear Euler equations become singular after a finite time.

In agreement with these observations and in the absence of boundary layers (full space or periodic domain), the following theorems are proved:

(i) Existence and uniqueness of the solution to the CNSE_ε for a finite time that depends on the size of the initial data, provided the initial data is smooth enough (say in H^s with $s > 3/2$). This time of existence is independent of ε and when ε goes to zero the solution converges to a solution of the compressible Euler equations.

(ii) Global (in time) existence of smooth solution[10] to the CNSE_ε provided the initial data is small enough with respect to ε.

These two points have their counterparts at the level of the Boltzmann equation:

(i) Existence and uniqueness (under stringent smallness assumptions) during a finite time independent of the Knudsen number, as proved by Nishida[13] (cf. also Caflisch[4]). When the Knudsen number goes to zero this solution converges to a local thermodynamic equilibrium solution governed by the compressible Euler equations.

(ii) Global existence for the solution to the Boltzmann equation provided the initial data is small enough with respect to the Knudsen number.

However, there are two other types of results concerning weak solutions. First, the global existence of weak solutions to the incompressible Navier–Stokes equations has been proved by Leray.[11] Second, using a method with many similarities to Leray's, DiPerna and Lions[8] have proved the global existence of a weak solution to a class of normalized Boltzmann equations, their so-called renormalized solution. This solution exists without assumptions concerning the size of the initial data with respect to the Knudsen number. Such a result also holds for the equation

$$\varepsilon\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon^q} C_{\mathrm{B}}(F_\varepsilon) \tag{73}$$

over a periodic spatial domain T^3.

The situation concerning the convergence to fluid dynamical limits (with ε going to zero) of solutions of the Boltzmann equation (73) with initial data of the form

$$F_\varepsilon = M(1 + \varepsilon^r g_\varepsilon) \tag{74}$$

continues to reflect this similarity. Following Nishida,[13] it can be shown that for smooth initial data (indeed very smooth) the solution of (73) is smooth for a time on the order of ε^{1-r}. For $r=1$ this time turns out to the independent of ε and durinig this time the solution converges (in the sense of the Theorem III) to the solution of the incompressible Euler equations when $q>1$ or to the solution of the incompressible Navier–Stokes equations when $q=1$. In this very case one obtains more precise information for the solution than the one given by ref. 7. It is the solution of the Cauchy problem itself, and not some (not so precisely defined) solution of the Boltzmann equation which converges. However, the price to pay lies in the much stronger hypothesis that must be made on the regularity of the initial data. For $r>1$ the solution is regular during a time that goes to infinity as ε vanishes; in this situation it converges to the solution of the linearized Navier–Stokes equations when $q=1$ or to the solution of the linearized Euler equation when $q>1$.

The borderline consists of the case $r=q=1$. In this case it is natural to conjecture that the DiPerna–Lions renormalized solutions of the Boltzmann equation converge (for all time and with no restriction on the size of the initial data) to a Leray solution of the incompressible Navier–Stokes equations. However, our proof of this result is incomplete without additional compactness assumptions.[2]

Leray's proof relies on the energy estimate

$$\tfrac{1}{2}\|u(t)\|^2 + \int_0^t \|\nabla_x u\|^2 \, ds \leqslant \tfrac{1}{2}\|u(0)\|^2 \tag{75}$$

For the Boltzmann equation the classical entropy estimate plays an analogous role in the proof of DiPerna and Lions. The entropy integrand can be modified by the addition of an arbitrary conserved density; the form chosen here is well suited for comparing F_ε with the absolute Maxwellian M:

$$\iint \left(F_\varepsilon(t) \log\left(\frac{F_\varepsilon(t)}{M}\right) - F_\varepsilon(t) + M \right) dv\, dx + \frac{1}{\varepsilon^{1+q}} \int_0^t \int D_\varepsilon \, dx\, ds$$
$$\leqslant \iint \left(F_\varepsilon(0) \log\left(\frac{F_\varepsilon(0)}{M}\right) - F_\varepsilon(0) + M \right) dv\, dx \tag{76}$$

where D_ε is the entropy dissipation term given by

$$D_\varepsilon = \frac{1}{4} \iiint (F'_{\varepsilon 1} F'_\varepsilon - F_{\varepsilon 1} F_\varepsilon) \log\left(\frac{F'_{\varepsilon 1} F'_\varepsilon}{F_{\varepsilon 1} F_\varepsilon}\right) b \, d\omega \, dv_1 \, dv$$

Here $F'_{\varepsilon 1}$, F'_ε, $F_{\varepsilon 1}$, and F_ε are understood to mean the evaluation of the distribution at the velocity values v'_1, v', v_1, and v, respectively.

The entropy integrand is a nonnegative, strictly convex function of F_ε with a quadratic minimum value of zero attained at $F_\varepsilon = M$. This suggests that in order to obtain bounds consistent with a formal expansion of the form (74) the initial data must be taken to satisfy the bound

$$\iint \left(F_\varepsilon(0) \log\left(\frac{F_\varepsilon(0)}{M}\right) - F_\varepsilon(0) + M \right) dv\, dx \leqslant K\varepsilon^{2r} \tag{77}$$

where K is an constant independent of ε.

The estimates (76) and (77) are enough to show that a sequence of functions g_ε in (74) is relatively compact in the weak topology of L^1 and that every weakly converging subsequence has a limit belonging to $\hat{L}^\infty(R_t^+, L^2(T^3) \otimes L^2_M)$. To obtain a strong convergence of the moments for the case $q = 1$, the above estimates are used with the averaging lemma[8,9] to obtain a compactness result for the integrand of the collision operator (72). These estimates are sufficient to show convergence to the linearized Navier–Stokes equations (42) for the case $r > q = 1$. However, for the case $q = r = 1$ an additional compactness assumption is needed to gain some time (weak) regularity for the moments $\langle vg_\varepsilon \rangle$. For $\varepsilon > 0$ these are not divergence-free functions and therefore (51) does not produces the classical estimates. Finally, to avoid concentration phenomena as $\varepsilon \to 0$, it is assumed that

$$(1 + |v|^2) \frac{g_\varepsilon^2}{(3 + \varepsilon g_\varepsilon)} M$$

lies in a weakly compact set of $L^1_{\text{loc}}(R_t^+, L^1(T^3) \otimes L^1_M)$. With these two assumptions one can show[2] the convergence (in the sense of Theorem III) of the velocity moments for renormalized solutions of the Boltzmann equation to a Leray solution of the incompressible Navier–Stokes equations.

Remark 6. The energy estimate (75) and the entropy estimate (76) are cornerstone estimates in the proofs of the global existence of the weak, or renormalized, solutions. This similarity can be emphasized by the observation that as $\varepsilon \to 0$ both sides of (76) converge to the corresponding sides of (75). A formal proof is easily obtained by an expansion of (76) in ε; the rigorous proof is given in ref. 2.

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